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ON THE STABILITY OF MOTION IN LINEAR APPROXIMATION

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Theorems on stability, uniform stability, asymptotic stability and uniform asymptotic stability are obtained by a uniform method for unsteady motions in the linear approximation, which generalize a number of results [1 — 6]. The theorems established are applied to one class of nonlinear systems, considered in [7].

1. Let us consider a system of differential equations of perturbed motion

$$\mathbf{x}^{\boldsymbol{\cdot}} = P(t)\mathbf{x} + \mathbf{X}(t, \mathbf{x}), \quad \mathbf{X}(t, 0) \equiv 0 \tag{1.1}$$

Here $P(t)$ is a matrix continuous for $t \geqslant 0$ and the function $\mathbf{X}(t, \mathbf{x})$ is continuous and satisfies the uniqueness conditions for the solution $\mathbf{x} = \mathbf{x}(t;\ t_0, \mathbf{x}_0)$ of system (1.1) in the domain

$$t \geqslant 0, \quad \|\mathbf{x}\| \leqslant H > 0 \tag{1.2}$$

The equations of linear approximation for system (1.1) are

$$\mathbf{x}^{\boldsymbol{\cdot}} = P(t)\mathbf{x} \tag{1.3}$$

By $U(t)$ we denote the fundamental matrix of the solutions of system (1.3) and by $K(t, t_0) = U(t)\,U^{-1}(t_0)$ we denote the Cauchy matrix.

Lemma 1. Assume that in the domain (1.2)

$$\|\mathbf{X}(t, \mathbf{x})\| \leqslant A(t)\|\mathbf{x}\|^m, \quad m > 1 \tag{1.4}$$

while the Cauchy matrix of system (1.3) satisfies the condition

$$\|K(t, t_0)\| \leqslant \varphi(t)\psi(t_0) \quad \text{for } t \geqslant t_0,\ t_0 \geqslant 0 \tag{1.5}$$

Here $A(\tau)$, $\varphi(\tau)$ and $\psi(\tau)$ are continuous functions positive for $\tau \geqslant 0$ Then the solutions of system (1.1) satisfy the inequality

$$\begin{aligned}&\|\mathbf{x}(t;\ t_0, \mathbf{x}_0)\| \leqslant \\ &\leqslant \varphi(t)\psi(t_0)\|\mathbf{x}_0\|\,[1 - (m-1)\psi^{m-1}(t_0)\|\mathbf{x}_0\|^{m-1}D(t_0, t)]^{-1/(m-1)}\end{aligned} \tag{1.6}$$

for all $t \geqslant t_0$, for which

$$(m-1)\psi^{m-1}(t_0)\|\mathbf{x}_0\|^{m-1}D(t_0, t) < 1 \tag{1.7}$$

$$D(t_0, t) = \int_{t_0}^{t} A(\tau)\varphi^m(\tau)\psi(\tau)\,d\tau$$

Proof. By the Cauchy formula [8]

$$\mathbf{x}(t; t_0, \mathbf{x}_0) = K(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} K(t, \tau)\mathbf{X}(\tau, \mathbf{x}(\tau; t_0, \mathbf{x}_0))\,d\tau$$

whence on the basis of (1.4) and (1.5) we obtain

$$\| \mathbf{x}(t; t_0, \mathbf{x}_0) \| \leqslant \varphi(t) \psi(t_0) \| \mathbf{x}_0 \| + \varphi(t) \int_{t_0}^{t} \psi(\tau) A(\tau) \| \mathbf{x}(\tau; t_0, \mathbf{x}_0) \|^m d\tau \tag{1.8}$$

From (1.8) it follows that

$$\frac{\| \mathbf{x}(t; t_0, \mathbf{x}_0) \|}{\varphi(t)} \leqslant \psi(t_0) \| \mathbf{x}_0 \| + \int_{t_0}^{t} A(\tau) \varphi^m(\tau) \psi(\tau) \left[\frac{\| \mathbf{x}(\tau; t_0, \mathbf{x}_0) \|}{\varphi(\tau)} \right]^m d\tau$$

Applying Bihary's lemma [9] (see [8], p. 112, Corollary 2) to the last inequality, we conclude that the estimate (1.6) is valid for all $t \geqslant t_0$ for which condition (1.7) is satisfied. Q. E. D.

2. We assume the fulfilment of conditions (1.4) and (1.5). Then the following assertions imply immediately from Lemma 1 (see inequalities (1.6) and (1.7)).

Theorem 1 (on stability in the linear approximation). Assume that: (1) for any $t_0 \geqslant 0$ there exists $N(t_0) > 0$, such that $\varphi(t) \psi(t_0) \leqslant N$ for all $t \geqslant t_0$ (*) (i.e., the function $\varphi(t)$ is bounded for all $t_0 \geqslant 0$);

(2) the condition

$$D(0, \infty) = \int_0^\infty A(\tau) \varphi^m(\tau) \psi(\tau) d\tau < \infty \tag{2.1}$$

is satisfied. Then the unperturbed motion of system (1.1) is stable.

Theorem 2 (on uniform stability in the linear approximation). Assume that: (1) there exist $N_0 > 0$ such that $\varphi(t) \psi(t_0) \leqslant N_0$ for all $t \geqslant t_0$ and all $t_0 \geqslant 0$ (**); (2) there exists $N_1 > 0$ such that

$$\psi^{m-1}(t_0) \int_{t_0}^\infty A(\tau) \varphi^m(\tau) \psi(\tau) d\tau \leqslant N_1 \quad \text{for all } t_0 \geqslant 0 \tag{2.2}$$

(note that (2.1) follows from (2.2)). Then the unperturbed motion of system (1.1) is stable uniformly with respect to t_0.

Theorem 3 (on asymptotic stability in the linear approximation). Assume that (1) condition (2.1) is satisfied; (2) $\lim \varphi(t) = 0$ as $t \to \infty$ (***). Then the unperturbed motion of system (1.1) is asymptotically stable.

Theorem 4. Assume that the conditions of Theorem 2 are satisfied and, in addition, $\lim \varphi(t) = 0$ as $t \to \infty$. Then the unperturbed motion of system (1.1) is stable uniformly with respect to t_0 and asymptotically stable uniformly with respect to $\mathbf{x}_0$.

Proof of Theorems 1-4. (1) For every $\varepsilon > 0$ and $t_0 \geqslant 0$ we set

$$\delta(\varepsilon, t_0) = \min \{ [2(m-1) \psi^{m-1}(t_0) D(t_0, \infty)]^{-1/(m-1)}, \varepsilon N^{-1} 2^{-1/(m-1)} \} > 0$$

If $\| \mathbf{x}_0 \| < \delta$, then from (1.6) and (1.7) it follows that $\| \mathbf{x}(t; t_0, \mathbf{x}_0) \| < \varepsilon$ for all $t \geqslant t_0$.

(2) In this case for each $\varepsilon > 0$ we can choose $\delta(\varepsilon) > 0$ independent of t_0

(3) From estimate (1.6) and condition 2) of Theorem 3 it follows that $\| \mathbf{x}(t; t_0, \mathbf{x}_0) \| \to 0$ as $t \to \infty$ if $\| \mathbf{x}_0 \| < \delta(\varepsilon, t_0)$.

(4) Theorem 4 follows from Theorems 2 and 3.

(*) The zero solution of system (1.3) is stable [8].

(**) The zero solution of system (1.3) is stable uniformly with respect to t_0 [8].

(***) The zero solution of system (1.3) is asymptotically stable.

Note. Theorems 1 and 3 were obtained by another method (and in somewhat different notation) in [6] for the case $(t) = A = \text{const}$.

Let us consider special cases of the theorems established for certain forms of the functions $\varphi(\tau)$, $\psi(\tau)$ and $A(\tau)$.

1°. $\varphi(\tau) = c_1 e^{-\alpha\tau}$, $\psi(\tau) = c_2 e^{(\alpha+\beta)\tau}$, $A(\tau) = c_3 e^{\gamma\tau}$, where c_1, c_2, c_3, α and β are positive constants and γ is a constant of arbitrary sign so that [1]

$$\| K(t, t_0) \| \leqslant B e^{-\alpha(t-t_0)} e^{\beta t_0}, \quad B = \text{const} \tag{2.3}$$

If the inequality (cf. [1])

$$(m-1)\alpha > \beta + \gamma \tag{2.4}$$

is satisfied, then integral (2.1) converges and by theorem 3 the unperturbed motion of system (1.1) is asymptotically stable. In particular, let $A(\tau) = A = \text{const}$ and, consequently, $\gamma = 0$; then for the asymptotic stability of the unperturbed motion of system (1.1) it is sufficient that the inequality (1, 2) $(m-1)\alpha > \beta$ be satisfied.

2° $\varphi(\tau) = c_1 \tau^{-\alpha}$, $\psi(\tau) = c_2 \tau^{\beta}$, $A(\tau) = c_3 \tau^{\gamma}$, $\tau \geqslant 1$, where c_1, c_2, c_3, α and β are positive constants of abitrary sign so that

$$\| K(t, t_0) \| \leqslant B t^{-\alpha} t_0^{\beta}, \quad B = \text{const}, \quad t \geqslant t_0 \geqslant 1 \tag{2.5}$$

If the condition

$$m\alpha - \gamma - \beta > 1 \tag{2.6}$$

is satisfied, then

$$\int_1^{\infty} A(\tau) \varphi^m(\tau) \psi(\tau) d\tau < \infty \tag{2.7}$$

and by Theorem 3 the unperturnbed motion of system (1.1) is asymptotically stable. In particular, for $A(\tau) = A = \text{const}$ ($\gamma = 0$) the condition (2.6) for asymptotic stability becomes

$$m\alpha - \beta > 1 \tag{2.8}$$

As Demidovich [3] showed, an estimate of the form (2.5) is satisfied by the Cauchy matrix of a fully regular system with nonpositive characteristic indices. In this connection inequality (2.8) coincides with asymptotic stability condition obtained in [3].

Note. If in 1° (respectively in 2°) $(m-1)\alpha > \beta$ (respectively, $m\alpha - \beta >$), then the constant γ can be positive; however, if $(m-1)\alpha \leqslant \beta$ (respectively, $m\alpha - \beta \leqslant 1$), then necessarily $\gamma < 0$; in the first case inequality (2.4) (respectively, (2.6)) determines the possible rate of growth of function $A(\tau)$ and in the second case, its necessary rate of decrease.

3. Let us consider the system [4]

$$x_s^{\cdot} = p(t) x_s + X_s(t, x_1, \ldots, x_n), \quad s = 1, \ldots, n \tag{3.1}$$

where $p(t)$ is a function continuous for $t \geqslant 0$ and the X_s satisfy condition (1.4). For system (3.1) we obviously have

$$\varphi(t) = \exp\left[\int_0^t p(\tau) d\tau\right], \quad \psi(t_0) = \exp\left[-\int_0^{t_0} p(\tau) d\tau\right]$$

On the basis of Theorems 1 and 3 we conclude that:

(1) if the conditions

$$\int_0^t p(\tau) d\tau \leqslant N = \text{const} \quad \text{for all} \quad t \geqslant 0 \tag{3.2}$$

$$\int_0^{\infty} A(t) \exp\left[(m-1)\int_0^t p(\tau) d\tau\right] dt < \infty \tag{3.3}$$

are satisfied, the unperturbed motion of system (3.1) is stable;

(2) if condition (3.3) is satisfied, in addition,

$$\lim_{t\to\infty} \int_0^t p(\tau) d\tau = -\infty \tag{3.4}$$

the unperturbed motion of system (3.1) is asymptotically stable.

The first of these statements was obtained by Persidskii [4] for $m = 2$, and $A(t) = A =$ const, while relation (3.3) has the form $\int_0^\infty \exp\left[\int_0^t p(\tau)\,d\tau\right] dt < \infty$

The question of the asymptotic stability of the zero solution of system (3.1) was not examined in [4].

Note. Let $A(t) = A =$ const; then condition (3.3) and (3.4) are satisfied for any $m > 1$ if the function

$$\exp\left[\int_0^t p(\tau)\,d\tau\right]$$

has a negative characteristic index.

Example. We consider system (3.1) with $p(t) = \sin\ln(1+t) + \cos\ln(1+t) - 2a$, $2a > 1$, while the X_s satisfy condition (1.4) with $A(t) = A =$ const. Since the characteristic index of the function

$$\exp\left[\int_0^t p(\tau)\,d\tau\right] = \exp\left[(1+t)\sin\ln(1+t) - 2at\right]$$

equals $1-2a < 0$, the unperturbed motion of system (3.1) is asymptotically stable. In this example the linear part of system (3.1) is not regular; however, neither Massera's criterion [5] (see also [8], p. 271) nor Malkin's generalized criterion [1, 2] are applicable here (thus, for $m = 2$ Massera's criterion is applicable only for $2a > 3$). It is interesting to note that the zero solution of the system

$$x_1^{\cdot} = -ax_1, \quad x_2^{\cdot} = [\sin\ln(1+t) + \cos\ln(1+t) - 2a]\,x_2 + x_1^2 \qquad (3.5)$$

is unstable for $1 < 2a < 1 + {}^1/_2 e^{-\pi}$ [11] (also see [2], pp. 368-369).

Note. The assertions of Theorems 1-4 cease to be true if integral (2.1) diverges, as is shown by example of the scalar equation $x^{\cdot} = -(1+t)^{-1}x + x^2$, whose solution

$$x(t) = \frac{x_0}{(1+t)[1 - x_0\ln(1+t)]}, \quad x(0) = x_0$$

leads to infinity in a finite time when $x_0 > 0$. In this example $m = 2$, $A(t) \equiv 1$, $\varphi(t) = (1+t)^{-1}$, $\psi(t_0) = 1 + t_0$ and

$$\int_0^\infty A(\tau)\,\varphi^m(\tau)\,\psi(\tau)\,d\tau = \int_0^\infty (1+\tau)^{-1}\,d\tau = \infty$$

4. Let us consider the case when the function X in Eqs. (1.1) satisfy the condition

$$\|X(t, x)\| \leqslant A(t)\,\|x\| \qquad (4.1)$$

(i.e., inequality (1.4) with $m = 1$). Using the Gronwall-Bellman Lemma [8, 12] analogously to Lemma 1 we can prove

Lemma 2. Assume that in domain (1.2) inequality (4.1) is satisfied and the Cauchy matrix satisfies conditon (1.5). Then the estimate

$$\|x(t; t_0, x_0)\| \leqslant \varphi(t)\,\psi(t_0)\,\|x_0\| \exp\left[\int_{t_0}^{t} A(\tau)\,\varphi(\tau)\,\psi(\tau)\,d\tau\right] \quad \text{for } t \geqslant t_0 \qquad (4.2)$$

is valid for the solution of system (1.1).

From Lemma 2 follows

Theorem 5 (1) if for any $t_0 \geqslant 0$ there exists $N(t_0) > 0$, such that

$$\varphi(t)\,\psi(t_0) \exp\left[\int_{t_0}^{t} A(\tau)\,\varphi(\tau)\,\psi(\tau)\,d\tau\right] \leqslant N \quad \text{for all} \quad t \geqslant t_0$$

the unperturbed motion of system (1.1) is stable;

(2) if in point 1) we can choose $N > 0$ independent of t_0, the unperturbed motion of system (1.1) is stable uniformly with respect to t_0,

(3) if $$\lim_{t\to\infty} \left\{\varphi(t)\,\psi(t_0) \exp\left[\int_{t_0}^{t} A(\tau)\,\varphi(\tau)\,\psi(\tau)\,d\tau\right]\right\} = 0$$

the unperturbed motion of system (1.1) is asymptotically stable.

Let us consider some special cases.

1° Let the zero solution of system (1.3) be stable uniformly with respect to t_0; and [8] the Cauchy matrix $K(t, t_0)$ is uniformly bounded for all $t \geqslant t_0$ and all $t_0 \geqslant 0$ consequently, we can set $\varphi(t) = \text{const}$ and $\psi(t_0) = \text{const}$. On the basis of point (2) of Theorem 5 we conclude that if

$$\int_0^\infty A(\tau)\,d\tau < \infty$$

the motion $\mathbf{x} = 0$ of system (1.1) is stable uniformly with respect to t_0. The result is close to those presented in [8, 12].

2° Let us assume that the zero solution of system (1.3) is exponentially asymptotically stable, i.e. $\| K(t, t_0)\| \leqslant Be^{-\alpha(t-t_0)}$, $B > 0$, $\alpha > 0$—const

and we can set $\varphi(t) = c_1 e^{-\alpha t}$, $\psi(\tau) = c_2 e^{\alpha\tau}$, $c_1 c_2 = B$; consequently

$$\varphi(t)\psi(t_0)\exp\left[\int_{t_0}^{t} A(\tau)\varphi(\tau)\psi(\tau)\,d\tau\right] = B\exp\left[\int_{t_0}^{t}(-\alpha + BA(\tau))\,d\tau\right] \tag{4.3}$$

On the basis of Theorem 5 we conclude (cf [5]):

(1) if for any $t_0 \geqslant 0$ there exists $N(t_0) > 0$, such that

$$\int_{t_0}^{t}[-\alpha + BA(\tau)]\,d\tau \leqslant N \text{ for all } t \geqslant t_0$$

the motion $\mathbf{x} = 0$ of system (1.1) is stable;

(2) if the number N can be chosen independent of t_0, the motion $\mathbf{x} = 0$ of system (1.1) is stable uniformly with respect to t_0;

(3) if

$$\int_{t_0}^{\infty}[-\alpha + BA(\tau)]\,d\tau = -\infty$$

the motion $\mathbf{x} = 0$ of system (1.1) is asymptotically stable.

In particular let $A(\tau) = A = \text{const}$; then from (4.2) and (4.3) we obtain

$$\| \mathbf{x}(t; t_0, \mathbf{x}_0)\| \leqslant B\|\mathbf{x}_0\| e^{-(\alpha - AB)(t-t_0)}$$

Consequently, if constant A is sufficiently small ($A < \alpha B^{-1}$) the motion $\mathbf{x} = 0$ of system is exponentially asymptotically stable [2].

5. In [7] we considered the system

$$\mathbf{y}^{\cdot} = Q(t)\mathbf{x} + R(t)\mathbf{y} + \mathbf{Y}^{\circ}(t, \mathbf{y}) + \mathbf{Y}(t, \mathbf{x}, \mathbf{y}) \tag{5.1}$$

$$\mathbf{x}^{\cdot} = P(t)\mathbf{x} + \mathbf{X}(t, \mathbf{x}, \mathbf{y}), \quad \mathbf{x} \in \mathbf{R}^n, \quad \mathbf{y} \in \mathbf{R}^k \tag{5.2}$$

under the assumption

$$\mathbf{Y}(t, 0, \mathbf{y}) \equiv 0, \ \mathbf{X}(t, 0, \mathbf{y}) \equiv 0$$

$$\frac{\|\mathbf{Y}(t, \mathbf{x}, \mathbf{y})\| + \|\mathbf{X}(t, \mathbf{x}, \mathbf{y})\|}{\|\mathbf{x}\|} \underset{t \geqslant 0}{\rightrightarrows} 0 \text{ for } \|\mathbf{x}\| + \|\mathbf{y}\| \to 0 \tag{5.3}$$

while the solutions of the system $\mathbf{x}^{*\cdot} = P(t)\mathbf{x}^*$ satisfy the condition

$$\| \mathbf{x}^*(t; t_0, \mathbf{x}_0^*)\| \leqslant B\|\mathbf{x}_0^*\| e^{-\alpha(t-t_0)} \quad (B > 0, \alpha > 0\text{-const}; t \geqslant t_0 \geqslant 0) \tag{5.4}$$

Hypothesis (3) of Theorem 1 in [7] is difficult to verify. Using the results of Sect. 2 this condition can not only be verified but also can reveal the asymptotic stability of the unperturbed motion $\|\mathbf{x}\| = \|\mathbf{y}\| = 0$ system (5.1). Let us assume that

$$\| Q(t)\| \leqslant M, \quad \| P(t)\| \leqslant M \quad \text{for } t \geqslant 0 \ (M = \text{const}) \tag{5.5}$$

and that the Cauchy matrix $K(t, t_0)$ of the linear system

$$\mathbf{y}^{\cdot} = R(t)\mathbf{y} \tag{5.6}$$

satisfies the conclusion (1.5). We consider the system

$$\mathbf{y}^{*\cdot} = R(t)\,\mathbf{y}^* + \mathbf{Y}^{\circ}(t, \mathbf{y}^*) \tag{5.7}$$

which is obtained from the first group of Eqs (5.1) with $\mathbf{x} = \mathbf{0}$ and whose solution we denote by $\mathbf{y}^*(t;\, t_0, \mathbf{y}_0^*)$. The variational equations for system (5.7) are

$$\xi^{\cdot} = \left[R(t) + \frac{\partial \mathbf{Y}^{\circ}(t, \mathbf{y}^*)}{\partial \mathbf{y}^*}\bigg|_{\mathbf{y}^*=\mathbf{y}^*(t;\, t_0, \mathbf{y}_0^*)}\right]\xi \tag{5.8}$$

By $\Omega(t;\, t_0, \mathbf{y}_0^*)$ we denote the fundamental matrix of solutions of system (5.8), $\Omega(t_0;\, t_0, \mathbf{y}_0^*) = E$, where E is the unit matrix. We assume that

$$\|\mathbf{Y}^{\circ}(t, \mathbf{y})\| \leqslant A(t)\,\|\mathbf{y}\|^m, \quad m > 1 \tag{5.9}$$

$$\left\|\frac{\partial \mathbf{Y}^{\circ}(t, \mathbf{y})}{\partial \mathbf{y}}\right\| \leqslant A_1(t)\,\|\mathbf{y}\|^{m-1} \tag{5.10}$$

If integral (2.1) converges, then by virtue of Lemma 1, for sufficiently small $\|\mathbf{y}_0^*\|$ for the solution of system (5.7) we have

$$\|\mathbf{y}^*(t;\, t_0, \mathbf{y}_0^*)\| \leqslant \varphi(t)\,\psi(t_0)\,\|\mathbf{y}_0^*\| \times \tag{5.11}$$

$$[1 - (m-1)\,\psi^{m-1}(t_0)\,\|\mathbf{y}_0^*\|^{m-1} D(t_0, \infty)]^{-1/(m-1)}$$

From (5.10) and (5.11) follows

$$\left\|\frac{\partial \mathbf{Y}^{\circ}(t, \mathbf{y}^*)}{\partial y^*}\bigg|_{\mathbf{y}^*=\mathbf{y}^*(t;\, t_0, \mathbf{y}_0^*)}\right\| \leqslant A_1(t)\,\varphi^{m-1}(t)\,G(t_0, \mathbf{y}_0^*) \tag{5.12}$$

$$G(t_0, \mathbf{y}_0^*) = \psi^{m-1}(t_0)\,\|\mathbf{y}_0^*\|^{m-1} \times$$

$$[1 - (m-1)\,\psi^{m-1}(t_0)\,\|\mathbf{y}_0^*\|^{m-1} D(t_0, \infty)]^{-1}$$

Applying Lemma 2 to system (5.8) and allowing for (5.12), we obtain

$$\|\Omega(t;\, t_0, \mathbf{y}_0^*)\| \leqslant \varphi(t)\,\psi(t_0)\exp[G(t_0, \mathbf{y}_0^*)\,D_1(t_0, t)]$$

$$D_1(t_0, t) = \int_{t_0}^{t} A_1(\tau)\,\varphi^m(\tau)\,\psi(\tau)\,d\tau$$

Assume that

$$D_1(0, \infty) = \int_0^{\infty} A_1(\tau)\,\varphi^m(\tau)\,\psi(\tau)\,d\tau < \infty \tag{5.13}$$

Then

$$\|\Omega(t;\, t_0, \mathbf{y}_0^*)\| \leqslant \varphi(t)\,\psi(t_0)\exp[G(t_0, \mathbf{y}_0^*)\,D_1(t_0, \infty)] \tag{5.14}$$

The following assertions stem from estimate (5.14):

(1) if $N_0 = \mathrm{const} > 0$ exists such that

$$\psi^{m-1}(t_0)\,D(t_0, \infty) \leqslant N_0, \; \psi^{m-1}(t_0)\,D_1(t_0, \infty) \leqslant N_0 \quad \text{for all} \quad t_0 \geqslant 0 \tag{5.15}$$

then we can find $h > 0$ for which

$$\|\Omega(t;\, t_0, \mathbf{y}_0^*)\| \leqslant L\varphi(t)\psi(t_0), \quad L = \mathrm{const} \tag{5.16}$$

follows from $\|\mathbf{y}_0^*\| \leqslant h$

(2) if, in addition, $N = \mathrm{const} > 0$, exists such that $\varphi(t)\,\psi(t_0) \leqslant N$ for all $t \geqslant t_0$ and all $t_0 \geqslant 0$ then

$$\|\Omega(t;\, t_0, \mathbf{y}_0^*)\| \leqslant NL \quad \text{for} \quad \|\mathbf{y}_0^*\| \leqslant h, t \geqslant t_0, \quad t_0 \geqslant 0 \tag{5.17}$$

(5.17) is th4 hypothesis of Theorem 1 in [7] that was difficult to verify.

Let conditions (5.15) be satisfied and consequently, inequality (5.16). Using, as in [7], the representation of the solutions of system (5.1) with the aid of the formulas [13]

$$\mathbf{y}(t;\, t_0, \mathbf{x}_0, \mathbf{y}_0) = \mathbf{y}^*(t;\, t_0, \mathbf{y}_0) + \int_{t_0}^{t} \Omega(t;\, \tau, \mathbf{y}(\tau;\, t_0, \mathbf{x}_0, \mathbf{y}_0)) \times$$

$$[Q(\tau)\,\mathbf{x}(\tau;\, t_0, \mathbf{x}_0, \mathbf{y}_0) + \mathbf{Y}(\tau, \mathbf{x}(\tau;\, t_0, \mathbf{x}_0, \mathbf{y}_0), \mathbf{y}(\tau;\, t_0, \mathbf{x}_0, \mathbf{y}_0))]\, d\tau$$

and taking into account the initial part of the proof of Theorem 1 in [7], as well as inequalities (5.11), (5.15) and (5.16), we obtain

$$\| \mathbf{y}(t; t_0, \mathbf{x}_0, \mathbf{y}_0) \| \leqslant \frac{\varphi(t)\psi(t_0)\|\mathbf{y}_0\|}{[1-(m-1)\psi^{m-1}(t_0)\|\mathbf{y}_0\|^{m-1}D(t_0)]^{1/(m-1)}} + \int_{t_0}^{t} L\varphi(t)\psi(\tau)C\|\mathbf{x}_0\|e^{-\gamma(\tau-t_0)}d\tau \leqslant \frac{\varphi(t)\psi(t_0)\|\mathbf{y}_0\|}{[1-(m-1)N_0\|\mathbf{y}_0\|^{m-1}]^{1/(m-1)}} +$$

$$CL\|\mathbf{x}_0\|\varphi(t)\int_{t_0}^{t}\psi(\tau)e^{-\gamma(\tau-t_0)}d\tau \quad (C>0,\ \gamma>0-\text{const})$$

From the preceding arguments follows

Theorem 6. Let there be given a system (5.1) satisfying conditions (5.2)-(5.6), (5.9) and (5.10). Assume that the Cauchy matrix of system (5.6) satisfies condition (1.5), the integrals (2.1) and (5.13) converge and inequalities (5.15) are satisfied. Then

1) for any $t_0 \geqslant 0$ there exists $F(t_0) > 0$, such that

$$\varphi(t)\psi(t_0) \leqslant F, \quad \varphi(t)\int_{t_0}^{t}\psi(\tau)e^{-\gamma(\tau-t_0)}d\tau \leqslant F \quad \text{for} \quad t \geqslant t_0$$

then the motion $\|\mathbf{x}\| = \|\mathbf{y}\| \overset{t_0}{=} 0$ of system (5.1) is Liapunov-stable and exponentially asymptotically $\mathbf{x}$-stable;

2) if in point 1) we can choose $F > 0$ not depending on t_0 then the Liapunov stability is uniform with respect to t_0;

3) if the conditions in point 1) are satisfied and

$$\lim_{t\to\infty}\varphi(t) = 0, \quad \lim_{t\to\infty}\left\{\varphi(t)\int_{t_0}^{t}\psi(\tau)e^{-\gamma(\tau-t_0)}d\tau\right\} = 0$$

then the motion $\|\mathbf{x}\| = \|\mathbf{y}\| = 0$ of system (5.1) is asymptotically stable and exponentially so relative to $\mathbf{x}$;

4) if the conditions in 2) and relations (5.18) are satisfied, then the motion $\|\mathbf{x}\| = \|\mathbf{y}\| = 0$ systems (5.1) is stable uniformly with respect to t_0, and asymptotically stable with respect to $\{\mathbf{x}_0, \mathbf{y}_0\}$ and exponentially so relative to $\mathbf{x}$.

Note. Points 1) and 3) of Theorem 6 give for system (5.1) results analogous to the "reduction principle" in Malkin's form [2] (with the sole difference that in the latter $A(t) = A = \text{const}$). However, Malkin's theorem ([2], page 383) is not applicable to system (5.1) since not all the conditions of this theorem are satisfied in the given case. First, the condition in [2] for the positive definiteness of the quadratic form (91.7) is not satisfied. As applied to system (5.1) this condition implies the sufficient smallness of $\|R(t)\|$ for all $t \geqslant 0$ (cf. [14]); meanwhile the elements of matrix $R(t)$ are not only assumed to be small but can even be unbounded. Secondly, the proof of the first two points of Malkin's first fundamental theorem on the critical cases ([2], Sect. 91) is in fact carried out under the assumption that the stability of the unperturbed motion of the "truncated" system is uniform with respect to t_0 independently of the terms of higher order than N and that the stability of the unperturbed motion of the "complete" system is uniform with respect to t_0 as well. The circumstance mentioned is explained by the fact that the number $\delta_1(h(\varepsilon), A)$, introduced on page 387 of [2] during the proof is assumed to be independent of t_0. If the two conditions listed are not satified, then, in the general case, the "reduction principle" does not hold (cf. [14]), as shown by the example of system (3.5) for which the zero solution of the truncated system $x_2^{\cdot} = [\sin \ln (1+t) + \cos \ln (1+t) - 2a]\, x_2$ is, according to the results of Sect. 3, stable independently of terms of order higher than the first (but not uniformly with respect to t_0).

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